

Distribution & Interpolation Spaces – Solution sheet 2

Exercise 1. The exercise is an application of Tonelli theorem. Denote by

$$E_f := \{(x, t) \in E \times [0, +\infty) : t < f(x)\}$$

the hypograph of the function f and observe that for every $x \in E$ it holds $\int_0^{f(x)} 1 dt = f(x)$. Therefore

$$\int_E f(x) dx = \int_E \left(\int_0^{f(x)} 1 dt \right) dx = \int_E \left(\int_0^{+\infty} \chi_{[0, f(x)]}(t) dt \right) dx = \int_{E \times [0, +\infty)} \chi_{E_f} dt dx,$$

where in the last equality we used Tonelli theorem. Again by Tonelli theorem,

$$\int_{E \times [0, +\infty)} \chi_{E_f} dt dx = \int_0^{+\infty} \left(\int_E \chi_{E_f}(t, x) dx \right) dt = \int_0^{+\infty} |\{x \in E : f(x) > t\}| dt.$$

This concludes the first part of the exercise, the second part follows the same computations with the only modification that

$$\int_0^{f(x)} p t^{p-1} dt = f(x)^p.$$

Exercise 2. Let $\{\rho_\epsilon\}_{\epsilon>0}$ be a standard family of mollifiers. Since $f \in L^1(\mathbb{R}^n)$, then we know that $f * \rho_\epsilon \rightarrow f$ in $L^1(\mathbb{R}^n)$ as $\epsilon \rightarrow 0$. Thus, up to subsequences, we can assume that

$$(f * \rho_\epsilon)(x) \rightarrow f(x) \quad \text{for a.e. } x \in \mathbb{R}^n, \text{ as } \epsilon \rightarrow 0. \quad (1)$$

By hypothesis we have that, for any $\epsilon > 0$,

$$(f * \rho_\epsilon)(x) = \int_{\mathbb{R}^n} f(y) \rho_\epsilon(x - y) dy = 0, \quad (2)$$

since $\rho_\epsilon \in C_c^\infty(\mathbb{R}^n)$. Putting together (1) and (2) we conclude that $f(x) = 0$ for almost every $x \in \mathbb{R}^n$.

Exercise 3. Let $\lambda \in \mathbb{R}$. We have

$$\begin{aligned} \|\lambda\varphi\|_{m,k} &= \max \{|D^\alpha(\lambda\varphi)| : |\alpha| \leq k, x \in K_{m+1} \setminus K_m\} \\ &= |\lambda| \max \{|D^\alpha(\varphi)| : |\alpha| \leq k, x \in K_{m+1} \setminus K_m\} \\ &= |\lambda| \|\varphi\|_{m,k}. \end{aligned}$$

Moreover, for every $\varphi_1, \varphi_2 \in C_c^\infty(\Omega)$

$$\begin{aligned} \|\varphi_1 + \varphi_2\|_{m,k} &= \max \{|D^\alpha(\varphi_1 + \varphi_2)| : |\alpha| \leq k, x \in K_{m+1} \setminus K_m\} \\ &\leq \max \{|D^\alpha(\varphi_1)| + |D^\alpha(\varphi_2)| : |\alpha| \leq k, x \in K_{m+1} \setminus K_m\} \\ &\leq \|\varphi_1\|_{m,k} + \|\varphi_2\|_{m,k}. \end{aligned}$$

Exercise 4. We need to show that the sets $D_{\varphi,k,\lambda}$ cover the whole space $C_c^\infty(\Omega)$ and that for every element in the intersection of two sets of the basis, there exists another set of the basis containing that element and such that it is itself contained in the intersection.

By choosing $\lambda \equiv 1, k \equiv 0$ we get that

$$\bigcup_{\varphi \in C_c^\infty(\Omega)} \{\psi \in C_c^\infty(\Omega) : \forall m \in \mathbb{N} \|\varphi - \psi\|_{m,0} < 1\} = C_c^\infty(\Omega).$$

Let now $D_{\varphi_1,k_1,\lambda_1}$ and $D_{\varphi_2,k_2,\lambda_2}$ be two elements of the basis with a nonempty intersection (otherwise it is trivial). Let $\varphi \in D_{\varphi_1,k_1,\lambda_1} \cap D_{\varphi_2,k_2,\lambda_2}$. Since $\forall m \in \mathbb{N} \|\varphi_1 - \varphi\|_{m,k_1(m)} < \lambda_1(m)$ and $\|\varphi_2 - \varphi\|_{m,k_2(m)} < \lambda_2(m)$, then for every $m \in \mathbb{N}$ there exist $\epsilon_1(m), \epsilon_2(m) > 0$ such that

$$\begin{aligned} \|\varphi_1 - \varphi\|_{m,k_1(m)} &< \lambda_1(m) - \epsilon_1(m) \\ \|\varphi_2 - \varphi\|_{m,k_2(m)} &< \lambda_2(m) - \epsilon_2(m). \end{aligned}$$

We now choose $k = \max(k_1, k_2)$ and $\lambda = \min(\epsilon_1, \epsilon_2)$. If $\psi \in D_{\varphi,k,\lambda}$ then

$$\begin{aligned} \|\varphi_1 - \psi\|_{m,k_1(m)} &\leq \|\varphi_1 - \varphi\|_{m,k_1(m)} + \|\varphi - \psi\|_{m,k_1(m)} \\ &< \lambda_1(m) - \epsilon_1(m) + \|\varphi - \psi\|_{m,k(m)} \\ &< \lambda_1(m) - \epsilon_1(m) + \lambda(m) \leq \lambda_1(m) \end{aligned}$$

and analogously

$$\|\varphi_2 - \psi\|_{m,k_2(m)} < \lambda_2(m) - \epsilon_2(m) + \lambda(m) \leq \lambda_2(m).$$

This shows that $D_{\varphi,k,\lambda} \subset D_{\varphi_1,k_1,\lambda_1} \cap D_{\varphi_2,k_2,\lambda_2}$.

Exercise 5. Suppose that $\exists K \subset \Omega$ such that $\text{spt } \varphi_n \subset K$ and $\varphi_n \rightarrow \varphi$ uniformly, together with all of its derivatives. Choose k and λ arbitrarily and let $D_{\varphi,k,\lambda}$ be an element of the basis given in the previous exercise. Since $K_m \uparrow \Omega$, then there exists $M \in \mathbb{N}$ such that $(K_{m+1} \setminus K_m) \cap K = \emptyset$ for all $m > M$. We define

$$\begin{aligned} \bar{k} &= \max_{m \leq M} k(m), \\ \bar{\lambda} &= \min_{m \leq M} \lambda(m). \end{aligned}$$

Since $\varphi_n \rightarrow \varphi$ uniformly (together with all of its derivatives), there exists $N \in \mathbb{N}$ such that for all $n > N$

$$\|\varphi_n - \varphi\|_{C^{\bar{k}}(\Omega)} < \bar{\lambda}.$$

This implies that $\forall n > N, \varphi_n \in D_{\varphi,k,\lambda}$. Note that one should also check that for any other set $D_{\psi,k,\lambda}$ containing φ , then the sequence is definitively contained in $D_{\psi,k,\lambda}$. However, this is an easy consequence of the fact that in exercise 4 we proved that those sets form a basis for the

topology. Indeed we proved the existence of a set $D_{\varphi, \bar{k}, \bar{\lambda}} \subset D_{\psi, k, \lambda} \cap D_{\varphi, k, \lambda}$ and from the previous computations we conclude that for some $N > 0$, the sequence $\varphi_n \in D_{\varphi, \bar{k}, \bar{\lambda}}$, for every $n > N$.

We first prove the existence of the compact set K such that $\text{spt } \varphi_n \subset K$, for all $n \in \mathbb{N}$. If by contradiction there is no K_M containing all the supports of φ_n , then the set

$$\Sigma = \{m \in \mathbb{N} : \exists n_m : \|\varphi_{n_m} - \varphi\|_{m,0} > 0\}$$

contains countably many integers. We can thus define

$$\begin{aligned} \bar{\lambda}(m) &= 1, & \text{if } m \in \mathbb{N} \setminus \Sigma \\ \bar{\lambda}(m) &= \frac{\|\varphi_{n_m} - \varphi\|_{m,0}}{2}, & \text{if } m \in \Sigma. \end{aligned}$$

Then we deduce that $\varphi_{n_m} \notin \mathcal{D}_{\varphi,0,\lambda}$ for all $m \in \Sigma$, which means that there are infinitely many elements of the sequence that are not contained in $\mathcal{D}_{\varphi,0,\lambda}$. This is of course in contradiction with the assumption that the sequence was topologically converging to φ . Thus we take $K = K_M$. Fix a multi-index α , a real number $\epsilon > 0$ and choose

$$k(m) = |\alpha| \quad \lambda(m) = \frac{\epsilon}{M},$$

$\forall m \in \mathbb{N}$. By hypothesis there exists $\bar{n} = \bar{n}(\epsilon) \in \mathbb{N}$ such that

$$\|\varphi_n - \varphi\|_{m,|\alpha|} < \frac{\epsilon}{M} \quad \forall n > \bar{n}, \forall m \in \mathbb{N},$$

from which we conclude that

$$\|D^\alpha \varphi_n - D^\alpha \varphi\|_{\text{Sup}(\Omega)} \leq \sum_{m=1}^{M-1} \|\varphi_n - \varphi\|_{m,|\alpha|} < \frac{\epsilon}{M}(M-1) < \epsilon.$$

Exercise 6 Suppose f is continuous and let U be a neighbourhood of $f(x)$. By definition of neighbourhood there is an open set $A \subset Y$ such that $f(x) \in A \subset U$: the open set $V = f^{-1}(A)$ is a neighbourhood of x and $f(V) \subset U$.

Conversely, suppose f is continuous at every point and let A be an open set in Y . If $x \in f^{-1}(A)$ then A is a neighbourhood of $f(x)$ and there is a neighbourhood V_x of x such that $f(V_x) \subset A$. This amounts to saying $V_x \subset f^{-1}(A)$. By definition there exists U_x open such that $x \in U_x$ and $U_x \subset V_x$. Therefore $f^{-1}(A)$ is an open set because it is union of open sets

$$f^{-1}(A) = \bigcup_{x \in f^{-1}(A)} U_x.$$